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The Complex Symmetry Gravitational Theory as a New Alternative of Dark Energy

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We propose that complex symmetry gravitational theory (CSGT) explain the accelerating expansion of universe. In this paper, universe is taken as the double complex symmetric space. Cosmological solution is obtained within CSGT. The conditions of the accelerating expansion of universe are discussed within CSGT. Moreover, the range of equation of state of matter ω_ε is given in the hyperbolic imaginary space.

KEY WORDS: complex symmetry gravitational theory; accelerating expansion of universe; dark energy.
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1. INTRODUCTION

One of the most surprising discoveries of the past decade is that expansion of the universe is currently speeding up rather than slowing down (Riess *et al.*, 1998). The case receives support from CMB (Spergel *et al.*, 2003) and the observations of supernovae and gravitational clustering (Perlmutter *et al.*, 1999; Peacock *et al.*, 2001). The accelerating expansion has been attributed to dark energy by which the universe is dominated. But, there is no direct evidence of dark energy and the nature of dark energy remains mysterious. At present, several candidate for dark energy have been presented. The simplest and most obvious candidate is cosmological constant (Sahni and Starobinsky, 2000). Although cosmological constant appears to satisfy all observations, the fine-tuning difficulties have prompted theorists to investigate a variety of alternative models where equation of state of dark energy is time dependent. Popular dark energy models include Quintessence (Sahni and Wang, 2000), Braneworld models (Sahni and Shtanov, 2003), Chaplygin gas (Kamenshchik *et al.*, 2001), Phantom energy (Caldwell, 2002) and modified gravity (Torres, 2002; Nojiri and Odintsov, 2003). Currently

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J.W. Moffat has proprosed that the nonsymmetric gravitational theory (NGT) (Einstein, 1956; Moffat, 1980; Moffat, 1988) explains the accelerating expansion of universe (Moffat, 2004), which is a new attempt. Nonsymmetry gravitational theory and complex symmetric gravitational theory (CSGT) (Moffat, 1957a, 1957b) are also generalized gravitational theory presented as the unified field theory of gravity and electromagnetism.

In the following, we shall apply CSGT to explain the accelerating expansion of universe. In this theory, universe is the double complex symmetry space. Our paper is organized as follows. We briefly review the foundations of CSGT in next section. In Section 3, the metric takes the canoncial Gaussian form for comoving coordinates. We solve Einstein's field equation without cosmological constant and obtain the cosmological solution. In Section 4, we investigate that when satisfying the same condition, matter in hyperbolic imaginary space gives rise to the accelerating expansion of universe. Furthermore, we give the range of equation of state of matter ω_ε in hyperbolic imaginary space. In Section 5, we formulate our conclusions.

2. THE FOUNDATION OF CSGT

In complex symmetry gravitational theory (CSGT), metric tensor is a complex symmetric tensor. Correspondingly, connection and curvature are forced to be complex. The real diffeomorphism symmetry of standard Riemannian geometry is extended to complex diffeomorphism symmetry.

We consider choosing an complex manifold of coordinates $\mathcal{M}_C^4$ and an complex symmetric metric defined by (Moffat, 2000; Wu Ya-bo, *et al.*, 2004; Wu Ya-bo, 1999).

$$g_{\mu\nu} = s_{\mu\nu} + Ja_{\mu\nu}, \tag{1}$$

where $s_{\mu\nu}$ and $a_{\mu\nu}$ are the real symmetric tensors, the double imaginary unit $J = i, \varepsilon$. The real contravariant tensor $s^{\mu\nu}$ is associated with $s_{\mu\nu}$ by the relation

$$s^{\mu\nu} s_{\mu\sigma} = \delta^\nu_\sigma, \tag{2}$$

and also

$$g^{\mu\nu} g_{\mu\sigma} = \delta^\nu_\sigma, \tag{3}$$

With the complex spacetime is also associated a complex symmetric connection

$$\Gamma^\lambda_{\mu\nu} = \Delta^\lambda_{\mu\nu} + J\Omega^\lambda_{\mu\nu}, \tag{4}$$

where $\Delta^\lambda_{\mu\nu}$ and $\Omega^\lambda_{\mu\nu}$ are also the real symmetric tensors. The complex symmetric connection $\gamma^\lambda_{\mu\nu}$ is determined by the equations

$$g_{\mu\nu;\lambda} = \partial_\lambda g_{\mu\nu} - g_{\rho\nu} g^\rho_{\mu\lambda} - g_{\mu\rho}\Gamma^\rho_{\nu\lambda} = 0, \tag{5}$$

Furthermore, we obtain the generalized curvature tensor

$$R^{\lambda}_{\mu\nu\sigma} = -\partial_\sigma \Gamma^{\lambda}_{\mu\nu} + \partial_\nu \Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\rho\nu}\Gamma^{\rho}_{\mu\sigma} - \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\rho}_{\mu\nu}, \tag{6}$$

and a contracted curvature tensor

$$R_{\mu\nu} := R^{\sigma}_{\mu\nu\sigma} = Q_{\mu\nu} + J\ P_{\mu\nu}, \tag{7}$$

where $Q_{\mu\nu}$ and $P_{\mu\nu}$ are the real symmetric tensors. From curvature tensor, we can obtain the four complex Bianchi identities

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right); \nu = 0, \tag{8}$$

The CSGT action is denoted by (Moffat, 1957a, 2000)

$$S = S_{\text{grav}} + S_M, \tag{9}$$

where S_{grav} and S_M are gravity action and matter action respectively.

We choose a following real action to guarantee a consistent set of field equations

$$S_{\text{grav}} = \frac{1}{2}\int d^4x[\mathcal{G}^{\mu\nu}R_{\mu\nu} + (\mathcal{G}^{\mu\nu}G_{\mu\nu})^{\dagger}], \tag{10}$$

and the matter part of action is

$$\frac{1}{\sqrt{-g}}\left(\frac{\delta S_M}{\delta g^{\mu\nu}}\right) = 8\pi G T_{\mu\nu}, \tag{11}$$

where $\mathcal{G}^{\mu\nu} := \sqrt{-g}g^{\mu\nu} = \mathcal{S}^{\mu\nu} + J\ \mathcal{A}^{\mu\nu}$, "$\dagger$" denotes complex conjugation, $T_{\mu\nu} = \tau_{\mu\nu} = \tau_{\mu\nu} + J\tau'_{\mu\nu}$ is a complex symmetric source tensor. The variation with respect to $g^{\mu\nu}$ yields the field equations

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = -8\pi G\mathcal{T}_{\mu\nu}, \tag{12}$$

where $\mathcal{R}_{\mu\nu} = \sqrt{-g}R_{\mu\nu}$, $\mathcal{R} = \mathcal{G}^{\mu\nu}R_{\mu\nu}$, $\mathcal{T}_{\mu\nu} = \sqrt{-g}T_{\mu\nu}$. Dividing the above equation by $\sqrt{-g}$ complex field Eq. (12) are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G\mathcal{T}_{\mu\nu}, \tag{13}$$

Equation (13) is written as

$$R_{\mu\nu} - 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{14}$$

3. COSMOLOGICAL SOLUTION AND MODIFIED FRIEDMANN EQUATION

Let us consider a real line element

$$dS^2 = -dt^2 + \alpha(r,t)dr^2 + \eta(r,t)(d\theta^2 + \sin^2\theta d\phi^2), \tag{15}$$

where $\alpha(r,t)$ and $\eta(r,t)$ are functions of real r and t. The complex symmetric tensor $g_{\mu\nu}$ is determined by

$$\begin{aligned} g_{00}(r,t) &= -(1+J),\\ g_{11}(r,t) &= \mu(r,t) = \alpha(r,t) + J\beta(r,t),\\ g_{22}(r,t) &= \gamma(r,t) = \eta(r,t) + J\xi(r,t),\\ g_{33}(r,t) &= \gamma(r,t)\sin^2\theta = [\eta(r,t) + J\xi(r,t)\sin^2\theta, \end{aligned} \tag{16}$$

Solving the $\Gamma^\lambda_{\mu\nu}$ and substituting into Eq. (6), we get

$$R_{00} = \frac{\mu_{tt}}{2\mu} - \frac{{\mu_t}^2}{4\mu^2} + \frac{\gamma_{tt}}{\gamma} - \frac{{\gamma_t}^2}{2\gamma^2}, \tag{17}$$

$$R_{11} = -\frac{\mu_{tt}}{2} + \frac{{\mu_t}^2}{4\mu} + \frac{\gamma_{rr}}{\gamma} - \frac{\gamma_t\mu_t}{2\gamma} - \frac{\mu_r\gamma_r}{2\mu\gamma} - \frac{\gamma_r^2}{2\gamma^2}, \tag{18}$$

$$R_{01} = \frac{\gamma_{tr}}{\gamma} - \frac{\mu_t\gamma_r}{2\mu\gamma} - \frac{\gamma_t\gamma_r}{2\gamma^2}, \tag{19}$$

where subscripts mean derivative with respect to t and r respectively.

In complex spacetime, the energy-momentum tensor takes

$$T^{\mu\nu} = [(\rho_C + p_C)U^\mu U^\nu + p_C g^{\mu\nu}] + J[(\rho_J + p_J)U'^\mu U'^\nu + p_J g^{\mu\nu}], \tag{20}$$

where ρ_C, $p_C(\rho_C, p_C > 0)$ and $\rho_J(\rho_J > 0)$, p_J are energy density and pressure respectively in real and imaginary spacetime. Moreover ρ_C, p_C are not variable with ρ_J, p_J. We define

$$s_{\mu\nu}U^\mu U^\nu = -1, \quad a_{\mu\nu}U'^\mu U'^\nu = -1, \tag{21}$$

$$T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}, \tag{22}$$

From Eqs. (3), (20) and (22), we get

$$T = (3p_C - \rho_C) - J^2(\rho_J + p_J) + J(4p_J), \tag{23}$$

If we assume by separation of variables

$$\begin{aligned} \mu(r,t) &= \alpha(r,t) + J\beta(r,t) = a^2(t)h(r) + Ja^2(t)\,h\,(r),\\ \gamma(r,t) &= \eta(r,t) + J\xi(r,t) = Y^2(t)r^2 + JY^2(t)r^2, \end{aligned} \tag{24}$$

Substituting into Eqs. (14) and (19), we obtain a special solution of equation $R_{01} = 0$

$$a(t) \approx Y(t), \tag{25}$$

Therefore, this gives rise to a metric of the form

$$dS^2 = -dt^2 + a^2(t)[h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{26}$$

This is the cosmological solution in CSGT. For instance, complex curvature R_{00} and R_{11} turn into

$$\begin{aligned} R_{00} = 3\frac{\ddot{a}}{a} &= -4\pi G[(3p_C + \rho_C) + J^2(p_J - \rho_J)] - J4\pi G[(p_C - \rho_C) \\ &\quad + (2 - J^2)\rho_J + (4 - J^2)p_J], \end{aligned} \tag{27}$$

$$\begin{aligned} R_{11} = (a\ddot{a}H + 2\dot{a}^2 h) + \frac{h_r}{rh} + J(a\ddot{a}h + 2\dot{a}^2 h) &= 4\pi Ga^2 h[(\rho_C - p_C) \\ + J^2(\rho_J - p_J)] + J4\pi Ga^2 h[(\rho_C - p_C) + J^2\rho_J + (J^2 - 2)p_J, \end{aligned} \tag{28}$$

And by the conservation law of energy momentum $T^{\mu\nu};\nu = 0$, the following equation is obtained

$$\begin{aligned} \dot{\rho}_C + \dot{p}_C + \frac{3\dot{a}}{a}(\rho_C + p_C) - \frac{\dot{p}_C}{1+J} + J\Big[(\dot{\rho}_J + \dot{p}_J) + \frac{3\dot{a}}{a}(\rho_J + p_J)\Big] \\ - J\frac{\dot{p}_J}{1+J} = 0, \end{aligned} \tag{29}$$

where dot means derivative with respect to time.

Assuming $h(r) = 1$, Eq. (26) is

$$dS^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\theta\phi^2)], \tag{30}$$

The line element Eq. (30) is the spatially-flat FRW metric. And the real and imaginary parts of complex curvature R_{00} and R_{11} are respectively

$$Q_{00} = \frac{3\ddot{a}}{a} = -4\pi g[(3p_C + \rho_C) + J^2(p_J - \rho_J)] \tag{27a}$$

$$P_{00} = 0 = 4\pi G[(p_C - \rho_C) + (2 - J^2)\rho_J + (4 - J^2)p_J], \tag{27b}$$

and

$$Q_{11} = 2\dot{a}^2 + a\ddot{a} = 4\pi Ga^2[(\rho_C - p_C) + J^2(\rho_J - p_J)], \tag{28a}$$

$$P_{11} = 2\dot{a}^2 + a\ddot{a} = 4\pi Ga^2[(\rho_C - p_C) + J^2\rho_J + (J^2 - 2)p_J], \tag{28b}$$

By calculating, we obtain $J = \varepsilon$ *i.e.*, the universe is the hyperbolic complex symmetry space. For instance, Eqs. (27a) and (28a) are

$$\ddot{a} = -\frac{4\pi G}{3} a[(3p_C + \rho_C) + (p_\varepsilon - \rho_\varepsilon)], \tag{31}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left[\rho_C + \frac{1}{2}(\rho_\varepsilon - p_\varepsilon)\right], \tag{32}$$

Equations (31) and (32) are modified Friedmann equations, where ρ_ε and p_ε are energy density and pressure in hyperbolic imaginary space. Moreover, the relation of ρ_C, p_C and ρ_ε, p_C is

$$p_C - \rho_C + \rho_\varepsilon + 3p_\varepsilon = 0, \tag{33}$$

4. THE ACCELERATING EXPANSIBLE UNIVERSE

In the section, we study the accelerating expansion of universe in the hyperbolic complex symmetry gravitational theory (HCSGT). The equation of state $\omega_\varepsilon = \frac{p_\varepsilon}{\rho_\varepsilon}$

Equations (31) and (32) are rewritten as

$$H^2 = \frac{8\pi G}{3}\left[\rho_C + \frac{1}{2}(\rho_\varepsilon - p_\varepsilon)\right] \tag{34}$$

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2} = \frac{\rho_C + 3p_C + (p_\varepsilon - \rho_\varepsilon)}{2\rho_C + (\rho_\varepsilon - p_\varepsilon)}, \tag{35}$$

From Eq. (34), we get $2\rho_C > p_\varepsilon$ for $H^2 > 0$. Since $q < 0$, we obtain

$$\rho_C + 3p_C - \rho_\varepsilon + p_\varepsilon < 0, \tag{36}$$

Furthermore, we investigate the condition of the accelerating expansion of universe if matter in the hyperbolic imaginary space is taken as a new alternative of dark energy.

If $p = 0$, Eq. (33) is $-\rho + 3p_\varepsilon + \rho_\varepsilon = 0$ *i.e.*, $\omega_\varepsilon > -\frac{1}{3}$. It satisfies the strong energy condition (SEC) $\omega \geq -\frac{1}{3}$ (Sahni, 2004). But dark energy must violate SEC in order to accelerate, *i.e.*, $\omega_\varepsilon < -\frac{1}{3}$ Substituting Eq. (33) into Eq. (36), we obtain $\omega_\varepsilon > -\frac{1}{2}$.

From the above analysis, we show if matter in the hyperbolic imaginary space is taken as a new alternative of dark energy in the HCSGT, the condition of the accelerating expansion of universe is

$$-\frac{1}{2} < \omega_\varepsilon < -\frac{1}{3}, \tag{37}$$

The result is acceptable and consistent with ref (Turner and White, 1997).

In the HCSGT, the conversion of energy momentum Eq. (29) turn into

$$\dot{\rho}_C + \frac{1}{2}\dot{p}_C + \frac{3\dot{a}}{a}(\rho_C + p_C) = 0$$

$$\dot{\rho}_J + \frac{1}{2}\dot{p}_J + \frac{3\dot{a}}{a}(\rho_J + p_J) = 0$$

Substituting into Eq. (34), we get

$$H^2 = \frac{8\pi G}{3}\left[(1+z)^{3(1+\omega_C)}e^{-\frac{1}{2}\int\frac{d(\rho_C\omega_C)}{d\rho_C}} + \frac{1}{2}(1+z)^{3(1+\omega_\varepsilon)}(1-\omega_\varepsilon)e^{-\frac{1}{2}\int\frac{d(\rho_\varepsilon\omega_\varepsilon\varepsilon)}{d\rho_\varepsilon}}\right] \tag{38}$$

5. CONCLUSIONS

In this paper we have proposed that CSGT may explain the accelerating expansion of universe. We concretely take a real line element and obtain the cosmological solution. Furthermore, conditions of the accelerating expansion of universe are discussed within HCSGT. Moreover, the equation of state of matter satisfies $-\frac{1}{2} < \omega_\varepsilon < -\frac{1}{3}$ if the matter in the hyperbolic imaginary space is taken as a new alternative of dark energy.

In above discussion, we have investigated the conditions of the accelerating expansion of universe within CSGT. But we don't deeply study the corresponding properties. These will be explored in further work.

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